IMAGE INTERPOLATION BASED ON OPTIMAL MASS PRESERVING MAPPINGS

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ABSTRACT

Time domain image interpolation, or image morphing, refers to a class of techniques for generating a set of smoothly changing intermediate images between two given images. Numerous methods have been proposed for this problem. In this note, we present a novel approach based on the theory of optimal mass transport.

1. INTRODUCTION

Image morphing is a class of techniques that deal with the metamorphosis of one image into another [6].

These techniques generate sequences of intermediate images in which an image gradually changes into another image over time. This methodology has also been referred to as *image interpolation,* which in this context means interpolation in the time domain and should be distinguished from image interpolation in the spatial domain. Image morphing has been widely used to create special effects in film and television. In this paper, we propose a novel approach to image interpolation.

There have been a number of different algorithms proposed for image interpolation. The mesh warping method [8] shows good distortion behavior, but has a critical drawback in that it requires the specification of features on the control mesh. These features may have an arbitrary structure, and the use of a user interface to define the feature correspondences may be difficult and time consuming. Field morphing as in [2] gives an easy-to-use and expressive method for specifying features. However, unexpected distortions referred to as "ghosts" appear, which means that a part of the initial image may show up in some unrelated part of the interpolated image. Lee *et al.* [6] proposed a method based on minimizing an energy functional. It guarantees the one-to-one property of the generated warp function, and so prevents the warped image from folding back upon itself. However, there is still a need to specify features. For a general review of interpolation methods, see [8] and the references therein.

Our approach does not require the specification of feature points and is thus a "blind" method. This method is based on the theory of optimal mass transport. The optimal mass transport problem was first formulated by the French mathematician Gaspar Monge in 1781 [7]. The original problem concerned finding the optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one place to another. The total amount of soil, i.e its the mass, was required to be conserved in the process. This problem was given a modern formulation by Kantorovich [5] and is also known as Monge-Kantorovich Problem (MKP).

2. OPTIMAL MASS TRANSPORT

We now give a modern formulation of the Monge-Kantorovich problem. Let Ω_0 and Ω_1 be two domains in $\mathbf{R}^d$, with smooth boundaries, each with a positive density function, μ_0 and μ_1, respectively. We assume

$$\int_{\Omega_0} \mu_0 = \int_{\Omega_1} \mu_1 \ , \tag{1}$$

so that the same total mass is associated with Ω_0 and Ω_1. Now we consider a class of diffeomorphisms u from Ω_0 to Ω_1 which map μ_0 to μ_1 in such a way that

$$\mu_0 = |Du| \mu_1 \circ u. \tag{2}$$

Here Du is the matrix of first derivatives of u. This "Jacobian equation" is a *mass preservation* (MP) constraint, and for a function satisfying (2) we will write $u \in$ MP.

A mapping u that satisfies this property may thus be thought of as defining a redistribution of a mass of material from one distribution μ_0 to another distribution μ_1. There may be many such mappings, and we want to pick out an optimal one in some sense. We define the L^p Kantorovich–Wasserstein metric as follows:

$$d_p(\mu_0, \mu_1)^p := \inf_{u \in \text{MP}} \int \|u(x) - x\|^p \mu_0(x) dx \tag{3}$$

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An optimal solution, when one exists [1], is a mapping $u \in \text{MP}$ that minimizes this integral.

The case $p = 2$ has been widely studied and will be the one used in our paper for image interpolation. Theoretical results [4] show that there is a unique optimal solution $\tilde{u} \in \text{MP}$, and that this unique solution is characterized as being the gradient of a convex function w, *i.e.*, $\tilde{u} = \nabla w$.

3. GENERAL MINIMIZATION PROBLEM

We now bstract the situation from the previous section. We assume that we have an initial mass-preserving map $u^0 : \Omega_0 \to \Omega_1$ which maps the density μ_0 to μ_1. We wish to deform it, *i.e.* to construct a family of maps $u^t : \Omega_0 \to \Omega_1$, indexed by time t, so as to minimize the "cost functional"

$$M[u^t] = \int_{\Omega_0} \Phi(u^t(x) - x)\mu_0(x)dx, \tag{4}$$

whilst preserving the mass preserving property (2). Here $\Phi : \mathbf{R}^d \to \mathbf{R}$ is a positive C^1 function. For example, we may take $\Phi(x) := \frac{\|x\|^2}{2}$ in which case this is the L^2 Monge-Kantorovich problem. We write u^t as the composition of two mappings

$$\begin{aligned} u^t &= u^0 \circ (s^t)^{-1}, \text{ or} && (5)\\ u^0 &= u^t \circ s^t && (6) \end{aligned}$$

where $s^t : \Omega_0 \to \Omega_0$ is a μ_0-mass preserving family of diffeomorphisms, *i.e.* satisfying

$$\mu_0 = |Ds^t|\mu_0 \circ s^t. \tag{7}$$

These diffeomorphisms s^t are generated by a vector field, the velocity field v^t, on Ω_0. Thus

$$\frac{\partial s^t}{\partial t} = v^t \circ s^t. \tag{8}$$

Since the maps s^t must preserve the density μ_0, the velocity field must satisfy

$$\text{div } (\mu_0 \, v^t) = 0. \tag{9}$$

By the chain rule applied to (6), the maps u^t satisfy

$$\frac{\partial u^t}{\partial t} + Du^t \cdot v^t = 0. \tag{10}$$

A change of variable argument shows that the cost decreases according to

$$\begin{aligned} \frac{dM}{dt} &= -\int \langle \nabla\Phi(u^0 - s), \frac{\partial s^t}{\partial t} \rangle \mu_0 \, dy \\ &= -\int \langle \nabla\Phi(u^t(x) - x), \zeta \rangle \, dx \end{aligned}$$

where $\zeta = \mu_0 \, v^t$. Using the Helmholtz decomposition, and imposing boundary conditions for the flow to remain in Ω_0, we take

$$\zeta = \nabla\Phi(u^t(x) - x) + \nabla p, \tag{11}$$

$$\text{div } (\zeta) = 0, \tag{12}$$

$$\zeta|_{\partial\Omega_0} \text{ is tangential to } \partial\Omega_0. \tag{13}$$

This leads to the following equation:

$$\frac{\partial u^t}{\partial t} = -\frac{1}{\mu_0} Du^t \cdot (\mathrm{I} - \nabla\Delta^{-1}\text{div })\nabla\Phi(u^t - \underline{\text{id}}), \tag{14}$$

where id denotes the identity map, and I is the identity matrix.

4. OPTIMAL IMAGE INTERPOLATION

We now propose the use of optimal mass transport to formulate a new method for image interpolation. The idea is to minimize a functional of the following form over mass-preserving mappings $u : \Omega_0 \to \Omega_1$:

$$M_\alpha[u] := \int (I_1 \circ u - I_0)^2 \; dx + \alpha^2 \int \|u(x) - x\|^2 \mu_0 \, dx, \tag{15}$$

for a fixed $\alpha \in \mathbf{R}$. Here the first term controls the "goodness of fit" between the (intensity) images $I_0 : \Omega_0 \to \mathbf{R}$ and $I_1 : \Omega_1 \to \mathbf{R}$, and the second Monge-Kantorovich term controls the the warping of the map. The function μ_0 is the mass density of the source image defined on Ω_0, and could be the same as I_0 or a smoothed version of I_0. Similarly, μ_1 is assumed to be the mass density of the target image defined on Ω_1. Using the results from Section 3, we derive the following descent equation for the optimum:

$$\begin{aligned} \frac{\partial u^t}{\partial t} &= \frac{-1}{\mu_0} Du^t \cdot (\mathrm{I} - \nabla\Delta^{-1}\text{div })[\frac{2}{\mu_0}(I_1 \circ u^t - I_0)\nabla I_0 \\ &\quad + \frac{1}{\mu_0^2}(I_1 \circ u^t - I_0)^2 \nabla\mu_0 + 2\alpha^2(u^t - \underline{\text{id}})]. \end{aligned} \tag{16}$$

5. IMPLEMENTATION AND EXAMPLES

We use standard techniques to solve Equation (16). In particular we have employed an upwinding scheme when computing Du^t, and the FFT when inverting the Laplacian on a rectangular grid. Standard centered differences were used for the other spatial derivatives. Once we numerically solve for the right hand side of (14), we use the result to update u^t.

We demonstrate now our image interpolation method with some examples. Let the starting and the ending images be denoted by I_0 and I_1 respectively, and the **optimal** mass transport function by u. In [3], it is shown that the optimal continuous family of image warping maps is simply given by

$$X^t(x) = (1-t)\,x + t\,u(x), \tag{17}$$

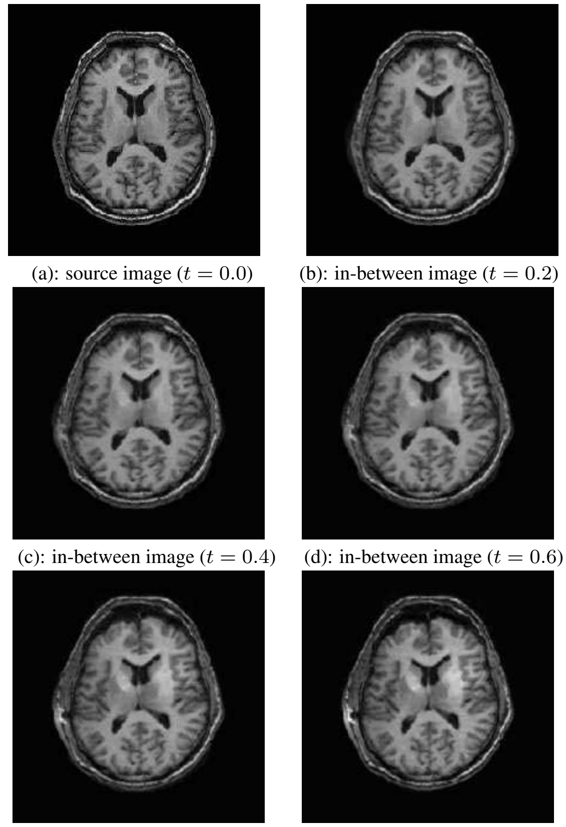
with a corresponding cross-dissolved image at time t given by

$$I^t(X^t(x)) = (1-t)\,I_0(x) + t\,I_1(u(x)). \tag{18}$$

Note that the warp function (17) is used to find a continuous transformation of the source image to target. One can always guarantee that the intermediate frames are mass-preserving simply by shading the target points according to $|DX^{-1}|\,\mu_0 \circ X^{-1}$.

Our first example is one in which we want to interpolate the motion of a *brain*. Figure 1(a) is a pre-operative T1 brain MRI image, and Figure 1(f) is another T1 brain MRI image from the same patient during surgery, after craniotomy and opening of the dura (Both of them are borrowed from *http://www-sop.inria.fr/epidaure/personnel/Olivier.Clatz*). We take Figure 1(a) and Figure 1(f) to be the starting and ending images, respectively. Figures 1(b) through 1(f) are the intermediate images generated at times $t = 0.2,\ 0.4,\ 0.6$ and 0.8 respectively.

The second example is an interpolation between two *flame* images. The original images are taken from a video clip by *Artbeats Digital Film Library*. The starting image, Figure 2, is the 24th frame in the sequence and the ending image, Figure 6, is the 29th frame in the sequence. Considering the frame rate of 30 frames/sec, there is about 0.2 seconds between these two images. Figures 3 through 5 present the interpolation result.



(a): source image ($t = 0.0$) (b): in-between image ($t = 0.2$)
(c): in-between image ($t = 0.4$) (d): in-between image ($t = 0.6$)
(e): in-between image ($t = 0.8$) (f): target image ($t = 1.0$)

Fig. 1. The morphing for *brain* images.

6. CONCLUSION AND FUTURE WORK

In this paper, we presented a natural method for image interpolation based on the classical problem of optimal mass transportation. We showed that for a L^2 version of the problem, one can derive easily-implementable gradient descent equations. Although applied here to the Monge–Kantorovich problem, the method used to enforce the mass preservation constraint is general and has other applications. In particular, we are currently working on using such constraints while minimizing standard Dirichlet-type energy functionals.

In particular, the concept of a harmonic mapping, defined as a minimizer of the Dirichlet integral, can be combined with a mass preservation constraint to obtain a new approach to mass-preserving diffeomorphisms [1]. We state the results for Euclidean space even though they apply more generally to Riemannian surfaces. As above, let $\Omega_0,\ \Omega_1 \subset \mathbf{R}^2$ be subdomains equipped with positive densities μ_0 and μ_1, respectively, and consider the minimization of the Dirichlet integral over all MP maps:

$$\min_{u \in MP} \int_{\Omega_0} \|Du\|^2. \tag{19}$$

A minimizer (when it exists) is called an area-preserving map of *minimal distortion.* Non-local and local gradient descent methods for computing such a map can be derived in a manner very similar to that described above for the Monge–Kantorovich functional [1]. These methods have applications to brain surface flattening and virtual colonoscopy. The numerical procedure and applications will be presented in a future paper. Future work will also include the use of Monge–Kantorovich type functionals as a regularizer in standard problems from computer vision such as optical flow.

7. REFERENCES

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Fig. 2. Starting image for *flame*.

Fig. 3. Intermediate image at t=0.2 for *flame*.

Fig. 4. Intermediate image at t=0.5 for *flame*.

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Fig. 5. Intermediate image at t=0.8 for *flame*.

Fig. 6. Ending image for *flame*.

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